

# Ramsey numbers of paths and graphs of the same order\*

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## Abstract

For graphs  $F_n$  and  $G_n$  of order  $n$ , if  $R(F_n, G_n) = (\chi(G_n) - 1)(n - 1) + \sigma(G_n)$ , then  $F_n$  is said to be  $G_n$ -good, where  $\sigma(G_n)$  is the minimum size of a color class among all proper vertex-colorings of  $G_n$  with  $\chi(G_n)$  colors. Given  $\Delta(G_n) \leq \Delta$ , it is shown that  $P_n$  is asymptotically  $G_n$ -good if  $\alpha(G_n) \leq \frac{n}{4}$ .

**Key Words:** Ramsey number; Ramsey goodness; Bounded degrees

## 1 Introduction

Let  $F$  and  $G$  be graphs. The Ramsey number  $R(F, G)$  is defined to be the smallest  $N$  such that any red-blue edge-coloring of  $K_N$  contains either a red  $F$  or a blue  $G$ . We shall write  $|G|$ ,  $\chi(G)$ ,  $\Delta(G)$  and  $\delta(G)$  as the order of  $G$ , the chromatic number, the maximum and minimum degrees of  $G$ , respectively. Denote by  $\sigma(G)$  the minimum size of a color class among all proper vertex-colorings of  $G$  with  $\chi(G)$  colors. Burr [3] had the following general bound.

**Lemma 1.** *For any graph  $G$ , if  $F$  is a connected graph with  $|F| \geq \sigma(G)$ , then*

$$R(F, G) \geq (\chi(G) - 1)(|F| - 1) + \sigma(G). \quad (1)$$

A connected graph  $F$  is said to be  $G$ -good if the equality in (1) holds, in which a  $K_p$ -good graph is said to be  $p$ -good in short. Chvátal [5] showed that a tree is  $p$ -good for any  $p$ . A family  $\mathcal{F}$  of graphs is said to be  $G$ -good if all large graphs in  $\mathcal{F}$  are  $G$ -good. Let  $P_n^k$  be the  $k$ th power of  $P_n$ , whose edges consist of pairs  $\{x, y\}$  with the distance in  $P_n$  at most  $k$ . Let  $bw(F)$  be the bandwidth of  $F$  of order  $n$ , which is the smallest integer  $k$  such that  $F$  is a subgraph of  $P_n^k$ . Burr and Erdős [4] showed that for any  $k$ , the family of connected graphs  $F$  with  $bw(F) \leq k$  is  $p$ -good for all  $p$ . Moreover, Nikiforov and Rousseau [12] showed that some larger families are  $p$ -good.

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Burr [3] proved that, for any fixed connected graph  $F$ , the class of graphs homeomorphic to  $F$  is *always-good*, i.e.,  $G$ -good for any fixed  $G$ . Using Blow-up Lemma [11], Allen, Brightwell and Skokan [1] proved that the family of connected graphs  $F$  with  $\Delta(F) \leq \Delta$  and  $bw(F) = o(|F|)$  is always-good, where  $\Delta$  is fixed.

Let us turn to  $R(F_n, G_n)$ , where  $F_n$  and  $G_n$  have the same order  $n$ . Burr [3] asked that if  $\Delta(G_n)$  is bounded and  $n$  is large, does

$$R(G_n, G_n) = (\chi(G_n) - 1)(n - 1) + \sigma(G_n)?$$

If the above equation holds, we say that  $G_n$  is *itself-good*. Thus  $P_n$  is itself-good shown by Gerencsér and Gyárfás [10], and  $C_n$  is itself-good shown by Faudree and Schelp [8], and Rosta [14]. However, it is difficult to estimate  $R(G_n, G_n)$  for general  $G_n$  even with  $\Delta(G_n)$  bounded, see [1, 6, 9].

Note that the general answer for Burr's question is negative as shown in [1] that  $R(P_n^k, P_n^k) \geq (k + 1)n - 2k$ . Recently, Pokrovskiy [13] proved  $R(P_n, P_n^k) = (n - 1)k + \lfloor \frac{n}{k+1} \rfloor$ , which solves a conjecture in [1]. For general  $G_n$ , it is difficult to answer whether  $P_n$  is  $G_n$ -good or not, so we ask the question in a weaker form: if  $\Delta(G_n)$  is bounded and  $n \rightarrow \infty$ , does

$$R(P_n, G_n) = (\chi(G_n) - 1)(n - 1) + \sigma(G_n) + o(n)?$$

The answer for the above question is positive in most cases. Let  $\alpha(G_n)$  be the independence number of  $G_n$ , which is at least the largest size of a color class in any proper vertex-coloring of  $G_n$ . It is easy to see that  $\sigma(G_n) \leq \frac{n}{\chi(G_n)}$  and  $\alpha(G_n) \geq \frac{n}{\chi(G_n)} \geq \frac{n}{\Delta(G_n)+1}$  for any graph  $G_n$ . A well-known fact [2] states that  $\alpha(G_n) = \frac{n}{\Delta(G_n)+1}$  if and only if  $G_n$  is a union of cliques of the same order, and thus we assume  $\Delta \geq 3$  in the following result to avoid trivial cases.

**Theorem 1.** *Let  $\Delta \geq 3$  be fixed, and  $G_n$  a graph of order  $n$  with  $\Delta(G_n) \leq \Delta$ . If  $n \rightarrow \infty$  and  $\alpha(G_n) \leq \frac{n}{4}$ , then*

$$R(P_n, G_n) = (\chi(G_n) - 1)n + \sigma(G_n) + o(n).$$

## 2 Lemmas

In the following context, we always assume that  $G_n$  is a graph of order  $n$  with  $\Delta(G_n) \leq \Delta$ . To simplify notations, we write  $k = \chi(G_n)$  and  $\sigma = \sigma(G_n)$  instead of  $k_n$  and  $\sigma_n$ , respectively. Let integers  $\Delta, n, N$  and  $\beta$ , and real  $\epsilon$  satisfy that

$$\Delta \geq 3, \quad 0 < \epsilon \leq \frac{1}{\Delta^5}, \quad \beta = \left\lceil \frac{1}{\epsilon^7} \right\rceil, \quad N = (k - 1)n + \sigma + \lceil \Delta^4 \epsilon n \rceil,$$

in which  $n$  is sufficiently large for such given  $\Delta$  and  $\epsilon$ .

For any red-blue edge-coloring of  $K_N$  on vertex set  $V$ , let  $R$  and  $B$  be the red and blue subgraph of the edge-colored  $K_N$  respectively, where  $\overline{R} = B$ . We shall prove that if  $R$  contains no red  $P_n$ , hence no  $C_m$  with  $m > n$ , then  $B$  contains  $G_n$ .

We will follow the method of [1] to embed  $G_n$  into the blue graph  $B$ . For some integer  $r$ , we call cycles

$$C^{(1)}, C^{(2)}, \dots, C^{(r)}$$

the ordered disjoint longest cycles in  $R$ , if  $C^{(i)}$  is a longest cycle in  $R \setminus \cup_{j=1}^{i-1} C^{(j)}$ ,  $1 \leq i \leq r$ . Denote by  $c_i$  be the length of  $C^{(i)}$ . Now we assume that  $r$  is the largest integer such that  $c_r \geq \epsilon^2 n$ . Then  $r \leq \frac{N}{\lceil \epsilon^2 n \rceil} < \frac{k}{\epsilon^2}$ . The following result is the Erdős-Gallai theorem for cycles [7], in which  $e(H)$  is the number of edges of  $H$ .

**Lemma 2.** *Let  $H$  be a graph, and  $c$  an integer with  $3 \leq c \leq |H|$ . Then either  $H$  contains a cycle of length at least  $c$  or*

$$e(H) < \frac{(c-1)(|H|-1)}{2} + 1.$$

We shall define an often used set  $W$  as follows.

$$W = V \setminus \cup_{j=1}^r V(C^{(j)}).$$

**Lemma 3.** *There is a subset  $W'$  of  $W$  with  $|W'| \geq (1-\epsilon)|W|$  such that every  $\Delta$  vertices of  $W'$  have at least  $|W'| - \Delta\epsilon n$  common blue-neighbors in  $W'$ .*

**Proof.** Let  $H$  be the subgraph of  $R$  induced by  $W$ . Lemma 2 with  $c = \lceil \epsilon^2 n \rceil$  implies  $e(H) < \epsilon^2 n |H|/2$ . Let  $W_0 \subseteq W$  such that any vertex in  $W_0$  has at least  $\epsilon n$  red-neighbors in  $W$ . Then  $|W_0| \leq \epsilon |W|$ . Let  $W' = W \setminus W_0$ . Then  $|W'| \geq (1-\epsilon)|W|$  and any vertex in  $W'$  has at least  $|W| - \epsilon n$  blue-neighbors in  $W$ . Hence every  $\Delta$  vertices in  $W'$  have at least  $|W'| - \Delta\epsilon n$  common blue-neighbors in  $W'$ .  $\square$

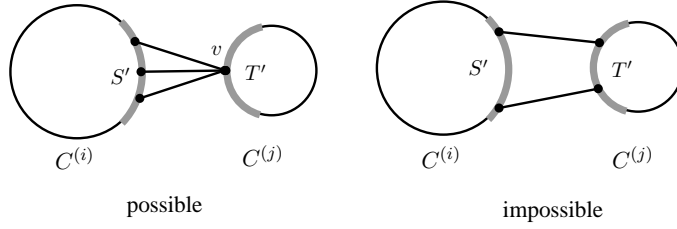
If  $|W| \geq n + 2\Delta\epsilon n$ , then  $|W'| > (1+\Delta\epsilon)n$ , and any  $\Delta$  vertices of  $W'$  in Lemma 3 have at least  $n$  common blue-neighbors in  $W'$ , and thus the subgraph of  $B$  induced by  $W'$  contains  $G_n$ . Therefore in the following proof we assume that

$$|W| < n + 2\Delta\epsilon n, \quad \text{hence} \quad r \geq k-1$$

as  $\frac{N-|W|}{n} > \frac{(k-2)n+\sigma(G)}{n} > k-2$ . This assumption will be used in the proof of Theorem 1. Let  $K_{n_1, \dots, n_r}$  be the complete  $r$ -partite graph with  $n_i$  vertices in the  $i$ th part. Note that  $n > c_1 \geq \dots \geq c_r \geq \epsilon^2 n$ . The integer  $\beta = \lceil \frac{1}{\epsilon^r} \rceil$  in the following result can be smaller, we assume so for further propose.

**Lemma 4.** *The blue graph  $B$  contains a  $K_{c_1-\beta, \dots, c_r-\beta}$  on partition  $(V_1, \dots, V_r)$  with  $V_i \subseteq V(C^{(i)})$  for  $1 \leq i \leq r$ .*

**Proof.** For any  $C^{(i)}$  and  $C^{(j)}$  with  $1 \leq i < j \leq r$ , we **claim** that there exist  $S \subseteq C^{(i)}$  and  $T \subseteq C^{(j)}$  with  $|S| \geq c_i - \frac{9}{\epsilon^4}$  and  $|T| \geq c_j - \frac{9}{\epsilon^4}$  such that  $S$  and  $T$  are completely adjacent in  $B$ . In fact, we partition each  $C^{(\ell)}$  into segments of consecutive vertices, in which each segment contains  $\lceil \frac{c_\ell}{2} \rceil$  vertices but at most one contains less. Let  $S'$  be a segment of  $C^{(i)}$ , and  $T'$  of  $C^{(j)}$ . Clearly, there are no two independent edges between  $S'$  and  $T'$  in  $R$ , since otherwise, we can enlarge  $C^{(i)}$  by adding the independent edges and edges of  $C^{(j)} \setminus T'$  as illustrated in Figure 1, impossible. Thus we can ignore at most one vertex in each of  $S'$  and  $T'$  such that the remaining vertices in  $S'$  and that in  $T'$  are completely adjacent in  $B$ .



**Figure 1** Possible and impossible edges between  $C^{(i)}$  and  $C^{(j)}$  in  $R$

Specifically for  $C^{(i)}$  and  $C^{(r)}$ , the later contains two segments  $T'_1$  and  $T'_2$  of consecutive vertices with  $|T'_2| \leq |T'_1| = \lceil \frac{c_r}{2} \rceil$ . By the mentioned process, we ignore one vertex in  $T'_1$  when considering it with a segment of  $C^{(i)}$ . As  $C^{(i)}$  has at most  $\lceil \frac{c_j}{|T'_1|} \rceil \leq \lceil \frac{2n}{c_r} \rceil \leq \frac{3}{\epsilon^2}$  such segments, we ignore at most  $\frac{3}{\epsilon^2}$  vertices in each of  $T'_1$  and  $C^{(i)}$  such that the remaining vertices in  $T'_1$  and that in  $C^{(i)}$  are completely adjacent in  $B$ . Same argument holds for  $T'_2$  and  $C^{(i)}$ . By considering each of  $C^{(1)}, \dots, C^{(r-1)}$  with  $C^{(r)}$ , we can ignore at most  $\frac{6}{\epsilon^2}$  vertices in each of  $C^{(1)}, \dots, C^{(r-1)}$  and at most  $\frac{6r}{\epsilon^2}$  vertices in  $C^{(r)}$  such that the remaining vertices in each of  $C^{(1)}, \dots, C^{(r-1)}$  and that in  $C^{(r)}$  are completely adjacent in  $B$ .

Repeat the process to  $C^{(1)}, \dots, C^{(i-1)}$  and  $C^{(i)}$  for each  $1 \leq i \leq r-1$ , and thus we can ignore at most  $\frac{9r}{\epsilon^4} \leq \beta$  vertices in each  $C^{(i)}$  such that the remaining vertices induce a subgraph of  $B$  that contains  $K_{c_1-\beta, \dots, c_r-\beta}$  as required.  $\square$

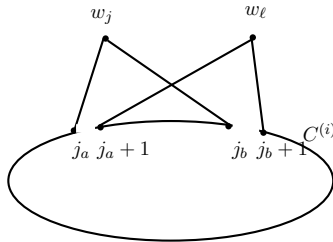
The following result says that the common blue-neighborhood of any  $\Delta$  vertices of  $W$  covers almost half vertices of each  $C^{(i)}$ , in which the idea of the proof comes from [3].

**Lemma 5.** *Let  $w_1, \dots, w_\Delta$  be vertices of  $W$ . Then the common blue-neighborhood  $\cap_{j=1}^\Delta N_B(w_j)$  contains at least  $\frac{1}{2}(c_i - \Delta^2)$  vertices of  $C^{(i)}$  for  $1 \leq i \leq r$ .*

**Proof.** Fix  $i$  with  $1 \leq i \leq r$ . Let vertices in  $V(C^{(i)})$  be labeled by  $\{1, 2, \dots, c_i\}$  clockwise. For  $1 \leq j \leq \Delta$ , let

$$Z_j = N_R(w_j) \cap V(C^{(i)}) = \{y_1, y_2, \dots\}, \quad \text{and} \quad Z_j + 1 = \{y_1 + 1, y_2 + 1, \dots\},$$

where  $Z_j + 1$  contains vertices of  $C^{(i)}$  next to that of  $Z_j$ . Clearly each vertex in  $Z_j + 1$  is non-adjacent to  $w_j$  in  $R$  from the maximality of  $C^{(i)}$ . For  $\ell \neq j$  with  $1 \leq \ell \leq \Delta$ , the vertex  $w_\ell$  has at most one red-neighbor in  $Z_j + 1$ , otherwise  $C^{(i)}$  could be enlarged as illustrated in Figure 2.



**Figure 2** a longer cycle

After considering  $w_1, \dots, w_\Delta$  in the same way, we know that  $(Z_j + 1) \cap (\cap_{\ell=1}^\Delta N_B(w_\ell))$  contains at least  $|Z_j + 1| - \Delta$  vertices. Inductively, we have

$$|(\cup_{j=1}^\Delta (Z_j + 1)) \cap (\cap_{\ell=1}^\Delta N_B(w_\ell))| \geq |\cup_{j=1}^\Delta (Z_j + 1)| - \Delta^2.$$

Let  $x = |\cap_{\ell=1}^\Delta N_B(w_\ell) \cap V(C^{(i)})|$ , the number of common blue-neighbors of  $w_1, \dots, w_\Delta$  in  $V(C^{(i)})$ . Then, we have  $c_i - x = |\cup_{j=1}^\Delta Z_j|$  and

$$x \geq |\cup_{j=1}^\Delta (Z_j + 1)| - \Delta^2 = |\cup_{j=1}^\Delta Z_j| - \Delta^2,$$

implying  $x \geq \frac{c_i - \Delta^2}{2}$ .  $\square$

Lemma 4 says that the cycles  $C^{(1)}, C^{(2)}, \dots, C^{(r)}$  are almost completely connected each other in  $B$ , and Lemma 5 tells us that edges between  $W$  and  $\cup_{\ell=1}^r C^{(\ell)}$  are dense in  $B$ .

### 3 Proof of the main results

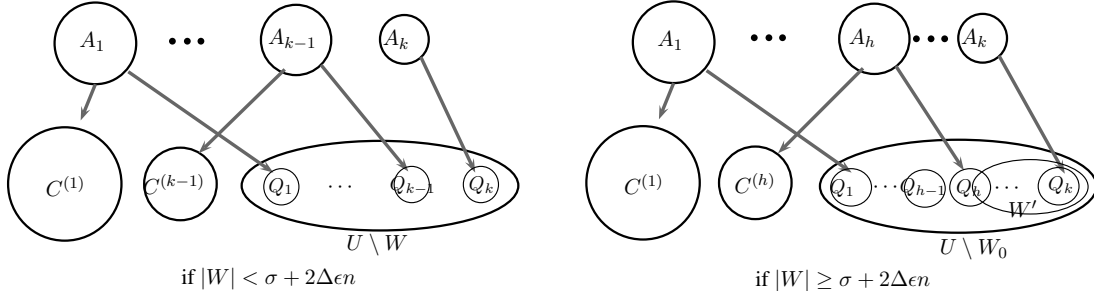
Recall  $W = V \setminus \cup_{j=1}^r V(C^{(j)})$  and  $r \geq k - 1$ , and define

$$U = V \setminus \cup_{j=1}^{k-1} V(C^{(j)}).$$

For disjoint subsets  $Q$  and  $Q'$  of  $U$ , we say that  $Q$  and  $Q'$  share none of  $C^{(k)}, C^{(k+1)}, \dots, C^{(r)}$ , if at least one of  $Q \cap V(C^{(i)})$  and  $Q' \cap V(C^{(i)})$  is empty for any  $k \leq i \leq r$ . When  $r = k - 1$ , any disjoint  $Q$  and  $Q'$  hold the property trivially. Let  $\{1, 2, \dots, k\}$  be the set of colors used for a proper vertex-coloring of  $G_n$ , and  $A_i$  the color class of vertices in color  $i$  such that  $a_1 \geq a_2 \geq \dots \geq a_k = \sigma$ , where  $a_i = |A_i|$  for  $1 \leq i \leq k$ . We shall embed  $A_1, A_2, \dots, A_k$  into  $B$ , in which if we choose a vertex from some  $C^{(i)}$  for a vertex of  $A_a$ , then we will not choose any from the same  $C^{(i)}$  for  $A_b$ , where  $a \neq b$ .

By Lemma 4, there is a  $K_{c_1-\beta, \dots, c_r-\beta}$  in  $B$  with partition  $(V_1, \dots, V_r)$  such that  $V_i \subseteq V(C^{(i)})$  for  $1 \leq i \leq r$ , and from Lemma 3, there is a subset  $W'$  of  $W$  with  $|W'| \geq (1 - \epsilon)|W| \geq |W| - 2\epsilon n$  such that every  $\Delta$  vertices in  $W'$  have at least  $|W'| - \Delta\epsilon n$  common blue-neighbors in  $W'$ . Let  $W_0 = W \setminus W'$ . Then  $|W_0| \leq 2\epsilon n$ . Let  $\bar{c}$  be the length of the largest cycle in  $U$ . Clearly,  $\bar{c} = c_k$  if  $r \geq k$  and  $\bar{c} < \epsilon^2 n$  otherwise.

**The Outline of the remaining proofs.** Partition  $U \setminus W$  or  $U \setminus W_0$  into  $Q_1, Q_2, \dots, Q_k$ , where some  $Q_i$  is possibly empty, such that any pair  $Q_i$  and  $Q_j$  share none of  $C^{(k)}, \dots, C^{(r)}$ ,  $A_k$  can be embedded into  $Q_k$ , and  $A_i$  can be embedded into  $V_i \cup Q_i$  for  $1 \leq i \leq k - 1$ . We will embed  $A_i$  into  $V_i \cup Q_i$  by an embedding  $\phi$  such that for every  $v \in A_i$  and its neighbors  $u_1, u_2, \dots$ ,  $\phi(v)$  is adjacent to  $\phi(u_1), \phi(u_2), \dots$  in  $B$ . The outline is illustrated by Figure 3.



**Figure 3** outline of the embedding

If  $|W| < \sigma + 2\Delta\epsilon n$ , we partition  $U \setminus W$  into  $Q_1, \dots, Q_k$  such that any pair of  $C^{(1)}, \dots, C^{(k-1)}$ ,  $Q_1, \dots, Q_k$  are almost completely connected in  $B$ . If  $|W| \geq \sigma + 2\Delta\epsilon n$ , we partition  $U \setminus W_0$  into  $Q_1, \dots, Q_{h-1}, Q_h, Q_{h+1}, \dots, Q_k$  for some  $h \leq k-1$  such that  $Q_1, \dots, Q_{h-1} \subseteq U \setminus W$ , and  $Q_{h+1}, \dots, Q_k \subseteq W'$ , and  $Q_h$  consists of remaining vertices in  $U \setminus W_0$ . We shall embed  $A_k$  into  $Q_k$  first. Then by Lemma 3, for each  $i$  with  $h \leq i \leq k-1$ , we shall embed a part of  $A_i$  into  $Q_i$  such that there are at most  $\Delta^2\epsilon n$  vertices left in  $W'$ . Then, as Lemma 3 and Lemma 5 says that any  $\Delta$  vertices in  $W'$  have at least  $(|V_i \cup Q_i| - r\Delta^2 - \Delta\epsilon n)/2$  common blue-neighbors in  $V_i \cup Q_i$ , we shall put the remaining vertices in  $A_i$  into  $V_i \cup Q_i$ , which will be explained in details.

Define

$$\eta_i = \left\lfloor \frac{c_i - 2\beta}{2} \right\rfloor$$

for  $1 \leq i \leq r$ . Lemma 4 and Lemma 5 imply that any  $\Delta$  vertices in  $W$  have at least  $\eta_i$  common blue-neighbors in  $V_i$  as  $\eta_i < \frac{1}{2}(c_i - \Delta^2 - \beta)$ . For  $1 \leq i \leq k-1$ , let us denote by

$$\begin{aligned} \Lambda &= \{j : 1 \leq j \leq k-1, a_j > \eta_j\}, \\ \Gamma &= \{j : 1 \leq j \leq k-1, a_j > c_j - \beta\}, \end{aligned}$$

and  $\lambda = |\Lambda|$ ,  $\gamma = |\Gamma|$ . Then  $\Gamma \subseteq \Lambda$ . If  $\Lambda = \emptyset$ , the proof is trivial as  $U$  is large enough for  $A_k$  and  $V_i$  is large enough for  $A_i$  for  $1 \leq i \leq k-1$ . So we assume that  $\Lambda \neq \emptyset$ .

Assume that  $p \in \Lambda$ . Then we have

$$\bar{c} \leq c_p \leq 2\eta_p + 3\beta < 2a_p + 3\beta \leq 2\alpha(G_n) + 3\beta \leq n/2 + 3\beta.$$

**Case 1.**  $|W| < \sigma + 2\Delta\epsilon n$ . As  $n \geq 2\alpha(G_n) + \bar{c} - 3\beta \geq a_i + \sigma + \bar{c} - 3\beta$  for  $1 \leq i \leq k-1$ , we have

$$\begin{aligned} |U \setminus W| + \sum_{i \in \Gamma} |V_i| &\geq N - (k-1-\gamma)n - (\sigma + 2\Delta\epsilon n) - r\beta \\ &\geq \gamma n + 2\epsilon n \geq \sum_{i \in \Gamma} (a_i + \bar{c}) + \sigma + \epsilon n. \end{aligned}$$

Partition  $U \setminus W$  into  $Q_1, \dots, Q_k$  such that any pair of them share none cycle of  $C^{(k)}, \dots, C^{(r)}$  as follows.

- For  $i \notin \Gamma \cup \{k\}$ , let  $Q_i = \emptyset$ .
- Note that  $|V_i| < c_i < a_i + r\beta$  for  $i \in \Gamma$  and  $c_r \leq \dots \leq c_k \leq \bar{c}$ , we can put a whole cycle among  $C^{(k)}, \dots, C^{(r)}$  one by one into  $Q_i$  until  $a_i + r\beta \leq |V_i \cup Q_i| \leq a_i + \bar{c} + r\beta$ . Then for  $Q_j$  with  $j \in \Gamma \setminus \{i\}$  that has not been constructed yet, we put a whole unused cycle among  $C^{(k)}, \dots, C^{(r)}$  one by one into  $Q_j$  similarly.
- Set  $Q_k = (U \setminus W) \setminus \cup_{i \in \Gamma} Q_i$ . Then  $|Q_k| \geq \sigma + r\beta$ .

Then the embedding can be constructed by Figure 3 and the explantation thereafter.

**Case 2.**  $|W| \geq \sigma + 2\Delta\epsilon n$ . As  $n \geq 2\alpha(G_n) + \bar{c} - 3\beta \geq 2a_i + \bar{c} - 3\beta$  for  $1 \leq i \leq k-1$ , we have

$$|U \setminus W_0| + \sum_{i \in \Lambda} |V_i| \geq N - (k-1-\lambda)n \geq \sum_{i \in \Lambda} (2a_i + \bar{c} + 2\Delta\epsilon n) + \sigma.$$

Partition  $U \setminus W_0$  into  $Q_1, \dots, Q_h, \dots, Q_k$  with  $1 \leq h \leq k-1$  such that  $Q_1, \dots, Q_{h-1} \subseteq U \setminus W$ ,  $Q_{h+1}, \dots, Q_k \subseteq W'$ , and any pair  $Q_i$  and  $Q_j$  share none cycle of  $C^{(k)}, \dots, C^{(r)}$  as follows.

- For  $i \notin \Lambda \cup \{k\}$ , let  $Q_i = \emptyset$ .
- Choose any  $\sigma + \Delta\epsilon n$  vertices in  $W'$  as  $Q_k$ .
- Let  $h$  be the smallest integer such that

$$|W'| \geq \sum_{\substack{j \in \Lambda \\ h < j < k}} (2a_j - |V_j|) + \sigma + (k-h)\Delta\epsilon n,$$

and if  $h = k-1$ , then  $|W'| \geq \sigma + \Delta\epsilon n$ . For each  $i \in \Lambda \cap \{h+1, \dots, k-1\}$ , choose any  $2a_i - |V_i| + \Delta\epsilon n$  vertices in  $W'$  as  $Q_i$ .

- For each  $i \in \Lambda \cap \{1, \dots, h-1\}$ , we put a whole cycle among  $C^{(k)}, \dots, C^{(r)}$  one by one into  $Q_i$  until  $2a_i + 2\Delta\epsilon n \leq |V_i \cup Q_i| \leq 2a_i + \bar{c} + 2\Delta\epsilon n$ . We can do so since  $c_i < 2\eta_i + 3\beta < 2a_i + 2\Delta\epsilon n$  for  $i \in \Lambda$  and  $c_r \leq \dots \leq c_k \leq \bar{c}$ .
- Let  $Q_h = (U \setminus W_0) \setminus \cup_{i \neq h} Q_i$ . Then  $|V_h \cup Q_h| \geq 2a_h + 2\Delta\epsilon n$ .

Then the embedding can be constructed by Figure 3 and the explantation thereafter.  $\square$

## References

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